

Bounds for canonical Green's function at cusps

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Abstract

In this article, we derive bounds for the canonical Green's function defined on a noncompact hyperbolic Riemann surface, when evaluated at two inequivalent cusps.

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Introduction

Notation and Main results Let X be a noncompact hyperbolic Riemann surface of finite volume $\text{vol}_{\text{hyp}}(X)$ with genus $g \geq 1$. Then, from the uniformization theorem from complex analysis, X can be realized as the quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \text{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting on the hyperbolic upper half-plane \mathbb{H} , via fractional linear transformations. Let \mathcal{P} denote the set of cusps of Γ . Put $\overline{X} = X \cup \mathcal{P}$.

The compact Riemann surface \overline{X} is embedded in its Jacobian variety $\text{Jac}(\overline{X})$ via the Abel-Jacobi map. Then, the pull back of the flat Euclidean metric by the Abel-Jacobi map is called the canonical metric, and the (1,1)-form associated to it is denoted by $\hat{\mu}_{\text{can}}(z)$. We denote its restriction to X by $\mu_{\text{can}}(z)$. Put

$$d_X = \sup_{z \in X} \frac{\mu_{\text{can}}(z)}{\mu_{\text{shyp}}(z)}.$$

The canonical Green's function is defined as the unique solution of the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c g_{\text{can}}(z, w) + \delta_w(z) = \mu_{\text{can}}(z),$$

with the normalization condition

$$\int_X g_{\text{can}}(z, w) \mu_{\text{can}}(z) = 0.$$

Let c_X denote a certain constant related to the Selberg zeta function (see equation (11) for definition). Let λ_1 denote the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} acting on smooth functions defined on X . Let $\kappa_p(z)$ denote the Kronecker's limit function associated to a cusp $p \in \mathcal{P}$, which is the constant term in the Laurent expansion of the Eisenstein series associated to the cusp $p \in \mathcal{P}$ at $s = 1$. Let $k_{p,q}(0) \in \mathbb{C}$ denote the zeroth Fourier coefficient of $\kappa_p(z)$ at the cusp $q \in \mathcal{P}$.

With notation as above, we have the following upper bound

$$\begin{aligned} |g_{\text{can}}(p, q)| &\leq 4\pi |k_{p,q}(0)| + \frac{2\pi}{g} \left(\sum_{\substack{s \in \mathcal{P} \\ s \neq p}} |k_{s,p}(0)| + \sum_{\substack{s \in \mathcal{P} \\ s \neq q}} |k_{s,q}(0)| \right) + \\ &\frac{1}{\text{vol}_{\text{hyp}}(X)} \left(\frac{4\pi(d_X + 1)^2}{\lambda_1} + \frac{|4\pi c_X|}{g} + \frac{43|\mathcal{P}|}{g} + 4\pi \right) + \frac{2 \log(4\pi)}{g}. \end{aligned}$$

Arithmetic significance Bounds for the canonical Green's function are very essential for calculating various arithmetic invariants like the Faltings height function and the Faltings delta function. Especially bounds for the canonical Green's function evaluated at two inequivalent cusps are essential for calculating the arithmetic self-intersection number of the dualizing sheaf defined on an arithmetic surface.

In [1], while bounding the arithmetic self-intersection number of the dualizing sheaf defined on the modular curve $X_0(N)$, A. Abbes and E. Ullmo computed bounds for the canonical Green's function evaluated at the cusps 0 and ∞ . In [11], H. Mayer has done the same for the modular curve $X_1(N)$.

Furthermore, in [12] U. Kühn also derived bounds for the arithmetic self-intersection number of the dualizing sheaf defined on any curve defined over a number field. Our bounds hold true for any noncompact hyperbolic Riemann surface of genus $g > 0$. So they can be directly used in [12], and we hope that this leads to better bounds for U. Kühn.

Lastly, using results from [2] and [3], our bounds can be easily extended to the case when X admits elliptic fixed points.

Organization of the paper In the first section, we set up our notation, introduce basic notions and recall some results. In the second section, we compute bounds for the canonical Green's functions evaluated at two inequivalent cusps.

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1 Background material

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane \mathbb{H} . Let X be the quotient space $\Gamma \backslash \mathbb{H}$, and let g denote the genus of X . The quotient space X admits the structure of a Riemann surface.

Let $\mathcal{P}(\Gamma)$ and $H(\Gamma)$ denote the set of parabolic and hyperbolic elements of Γ , respectively. Let \mathcal{P} be the finite set of cusps of X , respectively. Let \overline{X} denote $\overline{X} = X \cup \mathcal{P}$. Locally, away from the cusps, we identify \overline{X} with its universal cover \mathbb{H} , and hence, denote the points on $\overline{X} \setminus \mathcal{P}$ by the same letter as the points on \mathbb{H} .

Structure of \overline{X} as a Riemann surface The quotient space \overline{X} admits the structure of a compact Riemann surface. We refer the reader to section 1.8 in [13], for the details regarding the structure of \overline{X} as a compact Riemann surface. For the convenience of the reader, we recall the coordinate functions for the neighborhoods of cusps.

Let $p \in \mathcal{P}$ be a cusp and let $w \in U_r(p)$. Then $\vartheta_p(w)$ is given by

$$\vartheta_p(w) = e^{2\pi i \sigma_p^{-1} w},$$

where σ_p is a scaling matrix of the cusp p satisfying the following relations

$$\sigma_p i\infty = p \quad \text{and} \quad \sigma_p^{-1} \Gamma_p \sigma_p = \langle \gamma_\infty \rangle, \quad \text{where} \quad \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma_p = \langle \gamma_p \rangle$$

denotes the stabilizer of the cusp p with generator γ_p .

Hyperbolic metric We denote the (1,1)-form corresponding to the hyperbolic metric of X , which is compatible with the complex structure on X and has constant negative curvature equal

to minus one, by $\mu_{\text{hyp}}(z)$. Locally, for $z \in X \setminus \mathcal{E}$, it is given by

$$\mu_{\text{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{\text{Im}(z)^2}.$$

Let $\text{vol}_{\text{hyp}}(X)$ be the volume of X with respect to the hyperbolic metric μ_{hyp} . It is given by the formula

$$\text{vol}_{\text{hyp}}(X) = 2\pi(2g - 2 + |\mathcal{P}|).$$

The hyperbolic metric $\mu_{\text{hyp}}(z)$ is singular at the cusps, and the rescaled hyperbolic metric

$$\mu_{\text{shyp}}(z) = \frac{\mu_{\text{hyp}}(z)}{\text{vol}_{\text{hyp}}(X)}$$

measures the volume of X to be one.

Locally, for $z \in X$, the hyperbolic Laplacian Δ_{hyp} on X is given by

$$\Delta_{\text{hyp}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = -4y^2 \left(\frac{\partial^2}{\partial z \partial \bar{z}} \right).$$

Recall that $d = (\partial + \bar{\partial})$, $d^c = \frac{1}{4\pi i} (\partial - \bar{\partial})$, and $dd^c = -\frac{\partial \bar{\partial}}{2\pi i}$. Furthermore, we have

$$d_z d_z^c = \Delta_{\text{hyp}} \mu_{\text{hyp}}(z). \quad (1)$$

Canonical metric Let $S_2(\Gamma)$ denote the \mathbb{C} -vector space of cusp forms of weight 2 with respect to Γ equipped with the Petersson inner product. Let $\{f_1, \dots, f_g\}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. Then, the $(1,1)$ -form $\mu_{\text{can}}(z)$ corresponding to the canonical metric of X is given by

$$\mu_{\text{can}}(z) = \frac{i}{2g} \sum_{j=1}^g |f_j(z)|^2 dz \wedge d\bar{z}.$$

The canonical metric $\mu_{\text{can}}(z)$ remains smooth at the cusps, and measures the volume of X to be one. We denote the smooth $(1,1)$ -form defined by $\mu_{\text{can}}(z)$ on \bar{X} by $\hat{\mu}_{\text{can}}(z)$.

For $z \in X$, we put,

$$d_X = \sup_{z \in X} \frac{\mu_{\text{can}}(z)}{\mu_{\text{shyp}}(z)}. \quad (2)$$

As the canonical metric $\mu_{\text{can}}(z)$ remains smooth at the cusps and at the elliptic fixed points, and the hyperbolic metric is singular at these points, the quantity d_X is well-defined.

Canonical Green's function For $z, w \in X$, the canonical Green's function $g_{\text{can}}(z, w)$ is defined as the solution of the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c g_{\text{can}}(z, w) + \delta_w(z) = \mu_{\text{can}}(z), \quad (3)$$

with the normalization condition

$$\int_X g_{\text{can}}(z, w) \mu_{\text{can}}(z) = 0.$$

From equation (3), it follows that $g_{\text{can}}(z, w)$ admits a log-singularity at $z = w$, i.e., for $z, w \in X$, it satisfies

$$\lim_{w \rightarrow z} (g_{\text{can}}(z, w) + \log |\vartheta_z(w)|^2) = O_z(1). \quad (4)$$

Parabolic Eisenstein Series For $z \in X$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the parabolic Eisenstein series $\mathcal{E}_{\text{par},p}(z, s)$ corresponding to a cusp $p \in \mathcal{P}$ is defined by the series

$$\mathcal{E}_{\text{par},p}(z, s) = \sum_{\eta \in \Gamma_p \backslash \Gamma} \operatorname{Im}(\sigma_p^{-1} \eta z)^s.$$

The series converges absolutely and uniformly for $\operatorname{Re}(s) > 1$. It admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$, and the Laurent expansion at $s = 1$ is of the form

$$\mathcal{E}_{\text{par},p}(z, s) = \frac{1}{(s-1) \operatorname{vol}_{\text{hyp}}(X)} + \kappa_p(z) + O_z(s-1), \quad (5)$$

where $\kappa_p(z)$ the constant term of $\mathcal{E}_{\text{par},p}(z, s)$ at $s = 1$ is called Kronecker's limit function (see Chapter 6 of [5]). For $z \in X$, and $p, q \in \mathcal{P}$, the Kronecker's limit function $\kappa_p(\sigma_q z)$ satisfies the following equation (see Theorem 1.1 of [10] for the proof)

$$\kappa_p(\sigma_q z) = \sum_{n < 0} k_{p,q}(n) e^{2\pi i n \bar{z}} + \delta_{p,q} \operatorname{Im}(z) + k_{p,q}(0) - \frac{\log(\operatorname{Im}(z))}{\operatorname{vol}_{\text{hyp}}(X)} + \sum_{n > 0} k_{p,q}(n) e^{2\pi i n z}, \quad (6)$$

with Fourier coefficients $k_{p,q}(n) \in \mathbb{C}$. Let $p, q \in \mathcal{P}$, then for $z \in X$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the parabolic Eisenstein series $\mathcal{E}_{\text{par},p}(\sigma_q z, s)$ associated to $p \in \mathcal{P}$, admits a Fourier expansion of the form

$$\mathcal{E}_{\text{par},p}(\sigma_q z, s) = \delta_{p,q} y^s + \alpha_{p,q}(s) y^{1-s} + \sum_{n \neq 0} \alpha_{p,q}(n, s) W_s(nz), \quad (7)$$

where $\alpha_{p,q}(s)$, $\alpha_{p,q}(n, s)$, and $W_s(nz)$ the Whittaker function are given by equations (3.21), (3.22), and (1.37), respectively in [5].

Heat Kernels For $t \in \mathbb{R}_{>0}$ and $z, w \in \mathbb{H}$, the hyperbolic heat kernel $K_{\mathbb{H}}(t; z, w)$ on $\mathbb{R}_{>0} \times \mathbb{H} \times \mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t; z, w) = \frac{\sqrt{2} e^{-t/4}}{(4\pi t)^{3/2}} \int_{d_{\mathbb{H}}(z, w)}^{\infty} \frac{r e^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d_{\mathbb{H}}(z, w))}} dr,$$

where $d_{\mathbb{H}}(z, w)$ is the hyperbolic distance between z and w .

For $t \in \mathbb{R}_{>0}$ and $z, w \in X$, the hyperbolic heat kernel $K_{\text{hyp}}(t; z, w)$ on $\mathbb{R}_{>0} \times X \times X$ is defined as

$$K_{\text{hyp}}(t; z, w) = \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; z, \gamma w).$$

For $z, w \in X$, the hyperbolic heat kernel $K_{\text{hyp}}(t; z, w)$ satisfies the differential equation

$$\left(\Delta_{\text{hyp},z} + \frac{\partial}{\partial t} \right) K_{\text{hyp}}(t; z, w) = 0, \quad (8)$$

Furthermore for a fixed $w \in X$ and any smooth function f on X , the hyperbolic heat kernel $K_{\text{hyp}}(t; z, w)$ satisfies the equation

$$\lim_{t \rightarrow 0} \int_X K_{\text{hyp}}(t; z, w) f(z) \mu_{\text{hyp}}(z) = f(w). \quad (9)$$

To simplify notation, we write $K_{\text{hyp}}(t; z)$ instead of $K_{\text{hyp}}(t; z, z)$, when $z = w$.

For $t \in \mathbb{R}_{\geq 0}$ and $z \in X$, put

$$HK_{\text{hyp}}(t; z) = \sum_{\gamma \in \mathcal{P}(\Gamma)} K_{\mathbb{H}}(t; z, \gamma z).$$

The convergence of the above series follows from the convergence of the hyperbolic heat kernel $K_{\text{hyp}}(t; z)$ and the fact that $K_{\mathbb{H}}(t; z, \gamma z)$ is positive for all $t \in \mathbb{R}_{\geq 0}$, $z \in \mathbb{H}$, and $\gamma \in \Gamma$.

Selberg constant The hyperbolic length of the closed geodesic determined by a primitive non-conjugate hyperbolic element $\gamma \in \mathcal{H}(\Gamma)$ on X is given by

$$\ell_\gamma = \inf\{d_{\mathbb{H}}(z, \gamma z) \mid z \in \mathbb{H}\}.$$

For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the Selberg zeta function associated to X is defined as

$$Z_X(s) = \prod_{\gamma \in \mathcal{H}(\Gamma)} Z_\gamma(s), \quad \text{where } Z_\gamma(s) = \prod_{n=0}^{\infty} (1 - e^{(s+n)\ell_\gamma}).$$

The Selberg zeta function $Z_X(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$, with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian. Furthermore, $Z_X(s)$ has a simple zero at $s = 1$, and the following constant is well-defined

$$c_X = \lim_{s \rightarrow 1} \left(\frac{Z'_X(s)}{Z_X(s)} - \frac{1}{s-1} \right). \quad (10)$$

For $t \in \mathbb{R}_{\geq 0}$, the hyperbolic heat trace is given by the integral

$$H\operatorname{Tr} K_{\text{hyp}}(t) = \int_X HK_{\text{hyp}}(t; z) \mu_{\text{hyp}}(z).$$

The convergence of the integral follows from the celebrated Selberg trace formula. Furthermore, from Lemma 4.2 in [8], we have the following relation

$$\int_0^\infty (H\operatorname{Tr} K_{\text{hyp}}(t) - 1) dt = c_X - 1. \quad (11)$$

Automorphic Green's function For $z, w \in \mathbb{H}$ with $z \neq w$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, the free-space Green's function $g_{\mathbb{H},s}(z, w)$ is defined as

$$g_{\mathbb{H},s}(z, w) = g_{\mathbb{H},s}(u(z, w)) = \frac{\Gamma(s)^2}{\Gamma(2s)} u^{-s} F(s, s; 2s, -1/u),$$

where $u = u(z, w) = |z - w|^2 / (4 \operatorname{Im}(z) \operatorname{Im}(w))$ and $F(s, s; 2s, -1/u)$ is the hypergeometric function. There is a sign error in the formula defining the free-space Green's function given by equation (1.46) in [5], i.e., the last argument $-1/u$ in the hypergeometric function has been incorrectly stated as $1/u$, which we have corrected in our definition. We have also normalized the free-space Green's function defined in [5] by multiplying it by 4π .

For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the automorphic Green's function $g_{\text{hyp},s}(z, w)$ is defined as

$$g_{\text{hyp},s}(z, w) = \sum_{\gamma \in \Gamma} g_{\mathbb{H},s}(z, \gamma w).$$

The series converges absolutely uniformly for $z \neq w$ and $\operatorname{Re}(s) > 1$ (see Chapter 5 in [5]).

For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the automorphic Green's function satisfies the following properties (see Chapters 5 and 6 in [5]):

(1) For $\operatorname{Re}(s(s-1)) > 1$, we have

$$g_{\text{hyp},s}(z, w) = 4\pi \int_0^\infty K_{\text{hyp}}(t; z, w) e^{-s(s-1)t} dt.$$

(2) It admits a logarithmic singularity along the diagonal, i.e.,

$$\lim_{w \rightarrow z} (g_{\text{hyp},s}(z, w) + \log |\vartheta_z(w)|^2) = O_{s,z}(1).$$

(3) The automorphic Green's function $g_{\text{hyp},s}(z, w)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ with residue $4\pi/\text{vol}_{\text{hyp}}(X)$, and the Laurent expansion at $s = 1$ is of the form

$$g_{\text{hyp},s}(z, w) = \frac{4\pi}{s(s-1)\text{vol}_{\text{hyp}}(X)} + g_{\text{hyp}}^{(1)}(z, w) + O_{z,w}(s-1),$$

where $g_{\text{hyp}}^{(1)}(z, w)$ is the constant term of $g_{\text{hyp},s}(z, w)$ at $s = 1$.

(4) Let $p, q \in \mathcal{P}$ be two cusps. Put

$$C_{p,q} = \min \left\{ c > 0 \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_p^{-1} \Gamma \sigma_q \right\},$$

and $C_{p,p} = C_p$. Then, for $z, w \in X$ with $\text{Im}(w) > \text{Im}(z)$ and $\text{Im}(w)\text{Im}(z) > C_{p,q}^{-2}$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the automorphic Green's function admits the Fourier expansion

$$g_{\text{hyp},s}(\sigma_p z, \sigma_q w) = \frac{4\pi \text{Im}(w)^{1-s}}{2s-1} \mathcal{E}_{\text{par},q}(\sigma_p z, s) - \delta_{p,q} \log |1 - e^{2\pi i(w-z)}|^2 + O(e^{-2\pi(\text{Im}(w)-\text{Im}(z))}). \quad (12)$$

This equation has been proved as Lemma 5.4 in [5], and one of the terms was wrongly estimated in the proof of the lemma. We have corrected this error, and stated the corrected equation.

The space $C_{\ell,\ell\ell}(X)$ Let $C_{\ell,\ell\ell}(X)$ denote the set of complex-valued functions $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$, which admit the following type of singularities at finitely many points $\text{Sing}(f) \subset X$, and are smooth away from $\text{Sing}(f)$:

(1) If $s \in \text{Sing}(f)$, then as z approaches s , the function f satisfies

$$f(z) = c_{f,s} \log |\vartheta_s(z)| + O_z(1), \quad (13)$$

for some $c_{f,s} \in \mathbb{C}$.

(2) As z approaches a cusp $p \in \mathcal{P}$, the function f satisfies

$$f(z) = c_{f,p} \log (-\log |\vartheta_p(z)|) + O_z(1), \quad (14)$$

for some $c_{f,p} \in \mathbb{C}$.

Hyperbolic Green's function For $z, w \in X$ and $z \neq w$, the hyperbolic Green's function is defined as

$$g_{\text{hyp}}(z, w) = 4\pi \int_0^\infty \left(K_{\text{hyp}}(t; z, w) - \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) dt.$$

For $z, w \in X$ with $z \neq w$, the hyperbolic Green's function satisfies the following properties:

(1) For $z, w \in X$, we have

$$\lim_{w \rightarrow z} (g_{\text{hyp}}(z, w) + \log |\vartheta_z(w)|^2) = O_z(1). \quad (15)$$

(2) For $z, w \in X$, the hyperbolic Green's function satisfies the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c g_{\text{hyp}}(z, w) + \delta_w(z) = \mu_{\text{shyp}}(z), \quad (16)$$

with the normalization condition

$$\int_X g_{\text{hyp}}(z, w) \mu_{\text{hyp}}(z) = 0. \quad (17)$$

(3) For $z, w \in X$ and $z \neq w$, we have

$$g_{\text{hyp}}(z, w) = g_{\text{hyp}}^{(1)}(z, w) = \lim_{s \rightarrow 1} \left(g_{\text{hyp},s}(z, w) - \frac{4\pi}{s(s-1) \text{vol}_{\text{hyp}}(X)} \right). \quad (18)$$

The above properties follow from the properties of the heat kernel $K_{\text{hyp}}(t; z, w)$ (equations (8) and (9)) or from that of the automorphic Green's function $g_{\text{hyp},s}(z, w)$.

(4) From Proposition 2.4.1 in [4], (or from Proposition 2.1 in [2]) for a fixed $w \in X$, and for $z \in X$ with $\text{Im}(\sigma_p^{-1}z) > \text{Im}(\sigma_p^{-1}w)$, and $\text{Im}(\sigma_p^{-1}z) \text{Im}(\sigma_p^{-1}w) > C_p^{-2}$, we have

$$g_{\text{hyp}}(z, w) = 4\pi\kappa_p(w) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log(\text{Im}(\sigma_p^{-1}z))}{\text{vol}_{\text{hyp}}(X)} - \log \left| 1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)} \right|^2 + O(e^{-2\pi(\text{Im}(\sigma_p^{-1}z) - \text{Im}(\sigma_p^{-1}w))}), \quad (19)$$

i.e., for a fixed $w \in X$, as $z \in X$ approaches a cusp $p \in \mathcal{P}$, we have

$$g_{\text{hyp}}(z, w) = -\frac{4\pi \log(\text{Im}(\sigma_p^{-1}z))}{\text{vol}_{\text{hyp}}(X)} + O_{z,w}(1) = -\frac{4\pi \log(-\log|\vartheta_p(z)|)}{\text{vol}_{\text{hyp}}(X)} + O_{z,w}(1). \quad (20)$$

(5) For any $f \in C_{\ell, \ell\ell}(X)$ and for any fixed $w \in X \setminus \text{Sing}(f)$, from Corollary 3.1.8 in [4] (or from Corollary 2.5 in [2]), we have the equality of integrals

$$\int_X g_{\text{hyp}}(z, w) d_z d_z^c f(z) + f(w) + \sum_{s \in \text{Sing}(f)} \frac{c_{f,s}}{2} g_{\text{hyp}}(s, w) = \int_X f(z) \mu_{\text{shyp}}(z). \quad (21)$$

Certain Convergence results For $z \in \mathbb{H}$, put

$$P(z) = \sum_{\gamma \in \mathcal{P}(\Gamma)} g_{\mathbb{H}}(z, \gamma z).$$

The above series is invariant under the action Γ , and hence, defines a function on X . Furthermore, from Proposition 4.2.4 in [4] (or from 2.2 in [3]), the above series converges for all $z \in X$, and satisfies the following equation

$$P(z) = \sum_{p \in \mathcal{P}} \sum_{\eta \in \Gamma_p \setminus \Gamma} P_{\text{gen},p}(\eta z), \quad (22)$$

where $P_{\text{gen},p}(z) = \sum_{n \neq 0} g_{\mathbb{H}}(z, \gamma_p^n z)$.

Furthermore, from the absolute and uniform convergence of $P(z)$, and from that of the following series from Lemma 5.2 in [6]

$$\sum_{\gamma \in \mathcal{P}(\Gamma)} \Delta_{\text{hyp}} g_{\mathbb{H}}(z, \gamma z),$$

we get

$$\sum_{\gamma \in \mathcal{P}(\Gamma)} \Delta_{\text{hyp}} g_{\mathbb{H}}(z, \gamma z) = \Delta_{\text{hyp}} P(z) = \sum_{p \in \mathcal{P}} \sum_{\eta \in \Gamma_p \setminus \Gamma} \Delta_{\text{hyp}} P_{\text{gen},p}(\eta z), \quad (23)$$

$$\Delta_{\text{hyp}} P_{\text{gen},p}(z) = \sum_{n \neq 0} \Delta_{\text{hyp}} g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_{\infty}^n \sigma_p^{-1}z) = 2 \left(\frac{2\pi \text{Im}(\sigma_p^{-1}z)}{\sinh(2\pi \text{Im}(\sigma_p^{-1}z))} \right)^2 - 2. \quad (24)$$

For $z \in X$, put

$$H(z) = 4\pi \int_0^\infty \left(HK_{\text{hyp}}(t; z) - \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) dt. \quad (25)$$

The function $H(z)$ is invariant under the action of Γ , and hence, defines a function on X . Furthermore, from Proposition 4.3.2 (or from Proposition 2.9), it follows that $H(z)$ is well-defined on X , and for $z, w \in X$, we have

$$H(z) = \lim_{w \rightarrow z} (g_{\text{hyp}}(z, w) - g_{\mathbb{H}}(z, w)) - P(z).$$

From the above equation, for $z \in X$, we find

$$\Delta_{\text{hyp}} P(z) + \Delta_{\text{hyp}} H(z) = \Delta_{\text{hyp}} \lim_{w \rightarrow z} (g_{\text{hyp}}(z, w) - g_{\mathbb{H}}(z, w)).$$

For $z \in X$, since the integral

$$4\pi \int_0^\infty \left(K_{\text{hyp}}(t; z, z) - K_{\mathbb{H}}(t; 0) - \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) dt,$$

as well as the integral of the derivatives of the integrand are absolutely convergent, we can take the Laplace operator Δ_{hyp} inside the integral. So for $z \in X$, we find

$$\Delta_{\text{hyp}} P(z) + \Delta_{\text{hyp}} H(z) = 4\pi \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt. \quad (26)$$

From Lemma 5.2 and Proposition 7.3 in [9], for $z \in X$, we have the following relation

$$4\pi \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt = \sum_{\gamma \in \Gamma \setminus \{\text{id}\}} g_{\mathbb{H}}(z, \gamma z)$$

and the right-hand side of above equation remains bounded at the cusps. So we deduce that the left-hand side also remains bounded at the cusps.

From Proposition 4.3.3 in [4] (or from Proposition 2.10 in [3]), for $z \in X$ approaching a cusp $p \in \mathcal{P}$, we have

$$\begin{aligned} H(z) &= -\frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \log(\text{Im}(\sigma_p^{-1} z)) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} + 4\pi k_{p,p}(0) + O(\text{Im}(\sigma_p^{-1} z)^{-1}) \\ &= -\frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \log(-\log|\vartheta_p(z)|) + O_z(1). \end{aligned} \quad (27)$$

Hence, we can conclude that the function $H(z) \in C_{\ell, \ell}(X)$ with $\text{Sing}(f) = \emptyset$. Lastly, from equation (11), we have

$$\int_X H(z) \mu_{\text{hyp}}(z) = 4\pi(c_X - 1). \quad (28)$$

An auxiliary identity For notational brevity, put

$$C_{\text{hyp}} = \int_X \int_X g_{\text{hyp}}(\zeta, \xi) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \xi) dt \right) \mu_{\text{hyp}}(\xi) \mu_{\text{hyp}}(\zeta).$$

From Proposition 2.6.4 in [4] (or from Proposition 2.8 in [2]), for $z, w \in X$, we have

$$g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) = \phi(z) + \phi(w), \quad (29)$$

where from Corollary 3.2.7 in [4] (or from Remark 2.16 in [2]), the function $\phi(z)$ is given by the formula

$$\phi(z) = \frac{1}{2g} \int_X g_{\text{hyp}}(z, \zeta) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g^2}.$$

As $H(z) \in C_{\ell, \ell}(X)$, using the relations (1) and (28), and combining equation (26) with (21), we derive

$$\begin{aligned} \phi(z) &= \frac{H(z)}{2g} + \frac{1}{8\pi g} \int_X g_{\text{hyp}}(z, \zeta) \Delta_{\text{hyp}} P(\zeta) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g^2} - \int_X H(z) \mu_{\text{shyp}}(z) \\ &= \frac{H(z)}{2g} + \frac{1}{8\pi g} \int_X g_{\text{hyp}}(z, \zeta) \Delta_{\text{hyp}} P(\zeta) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g^2} - \frac{2\pi(c_X - 1)}{g \text{vol}_{\text{hyp}}(X)}. \end{aligned} \quad (30)$$

For more details regarding the proof for the above computation, we refer the reader to Theorem 4.3.8 in [4] (or Corollary 2.12 in [3]).

Key identity For $z \in X$, we have the relation of differential forms

$$g \mu_{\text{can}}(z) = \left(\frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) \mu_{\text{hyp}}(z) + \frac{1}{2} \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z).$$

This relation has been established as Theorem 3.4 in [6], when X is compact, which easily extends to our case. Furthermore, from Corollary 3.2.5 in [4] (or from Corollary 2.15 in [2]), for any $f \in C_{\ell, \ell\ell}(X)$, we have

$$g \int_X f(z) \mu_{\text{can}}(z) = \left(\frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) \int_X f(z) \mu_{\text{hyp}}(z) + \frac{1}{2} \int_X f(z) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z). \quad (31)$$

2 Bounds for canonical Green's functions at cusps

Let $p, q \in \mathcal{P}$ be two cusps with $p \neq q$. Then, from equation (29), we find

$$g_{\text{can}}(p, q) = \lim_{z \rightarrow p} \lim_{w \rightarrow q} (g_{\text{hyp}}(z, w) - \phi(z) - \phi(w)). \quad (32)$$

From equations (19) and (27), we know the asymptotics of the functions $g_{\text{hyp}}(z, w)$ and $H(z)$ at the cusps, respectively. So if we can compute the asymptotics of the integral

$$\int_X g_{\text{hyp}}(z, \zeta) \Delta_{\text{hyp}} P(\zeta) \mu_{\text{hyp}}(\zeta)$$

at the cusps, we will be able to compute an upper bound for the canonical Green's function when evaluated at two different cusps.

For the remaining part of the thesis, for $p \in \mathcal{P}$ a cusp and $z \in \mathbb{H}$, we denote $\text{Im}(\sigma_p^{-1}z)$ by y_p .

In the following two lemmas, we compute the zeroth Fourier coefficient of the automorphic Green's function and the hyperbolic Green's function.

Lemma 2.1. *Let $p, q \in \mathcal{P}$ be two cusps. Then, for z and $w = u + iv \in X$ with $y_p > v$ and $vy_p > 1$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have*

$$\int_0^1 g_{\text{hyp}, s}(z, \sigma_q w) du = \frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par}, q}(z, s) + \frac{4\pi \delta_{p, q}}{2s-1} (v^s y_p^{1-s} - v^{1-s} y_p^s). \quad (33)$$

Furthermore, for $v > y_p$ and $vy_p > 1$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have

$$\int_0^1 g_{\text{hyp}, s}(z, \sigma_q w) du = \frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par}, q}(z, s). \quad (34)$$

Proof. For z and $w = u + iv \in X$ with $y_p > v$ and $vy_p > 1$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, combining Lemmas 5.1 and 5.2 of [5], we have

$$\int_0^1 g_{\text{hyp}, s}(z, \sigma_q w) du = \frac{4\pi y_p^{1-s}}{2s-1} (\delta_{p, q} v^s + \alpha_{p, q}(s) v^{1-s}) + \frac{4\pi v^{1-s}}{2s-1} \sum_{n \neq 0} \alpha_{p, q}(n, s) W_s(n \sigma_p^{-1} z).$$

The expression on the right-hand side of the above equation can be rewritten as

$$\frac{4\pi v^{1-s}}{2s-1} \left(\delta_{p, q} y_p^s + \alpha_{p, q}(s) y_p^{1-s} + \sum_{n \neq 0} \alpha_{p, q}(n, s) W_s(n \sigma_p^{-1} z) \right) + \frac{4\pi \delta_{p, q}}{2s-1} (v^s y_p^{1-s} - v^{1-s} y_p^s). \quad (35)$$

For $s \in \mathbb{C}$ and $\text{Re}(s) > 1$, from the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{\text{par}, q}(z, s)$ described in equation (7), we get

$$\frac{4\pi v^{1-s}}{2s-1} \left(\delta_{p, q} y_p^s + \alpha_{p, q}(s) y_p^{1-s} + \sum_{n \neq 0} \alpha_{p, q}(n, s) W_s(n \sigma_p^{-1} z) \right) = \frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par}, q}(z, s). \quad (36)$$

Combining equations (35) and (36) proves equation (33).

For $v > y_p$ and $vy_p > 1$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, combining Lemmas 5.1 and 5.2 of [5], we have

$$\int_0^1 g_{\text{hyp},s}(z, \sigma_q w) du = \frac{4\pi v^{1-s}}{2s-1} \left(\delta_{p,q} y_p^s + \alpha_{p,q}(s) y_p^{1-s} + \sum_{n \neq 0} \alpha_{p,q}(n, s) W_s(n \sigma_p^{-1} z) \right).$$

From equation (36), we derive that

$$\int_0^1 g_{\text{hyp},s}(z, \sigma_q w) du = \frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par},q}(z, s),$$

which proves equation (34), and completes the proof of the lemma. \square

Lemma 2.2. *Let $p, q \in \mathcal{P}$ be two cusps. Then, for z and $w = u + iv \in X$ with $y_p > v$ and $vy_p > 1$, we have*

$$\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du = 4\pi \kappa_q(z) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\operatorname{vol}_{\text{hyp}}(X)} + 4\pi \delta_{p,q}(v - y_p). \quad (37)$$

Furthermore, for $v > y_p$ and $vy_p > 1$, we have

$$\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du = 4\pi \kappa_q(z) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\operatorname{vol}_{\text{hyp}}(X)}. \quad (38)$$

Proof. Observe that

$$\begin{aligned} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) du &= \int_0^1 \lim_{s \rightarrow 1} \left(g_{\text{hyp},s}(z, \sigma_q w) - \frac{4\pi}{s(s-1) \operatorname{vol}_{\text{hyp}}(X)} \right) du = \\ &= \lim_{s \rightarrow 1} \left(\int_0^1 g_{\text{hyp},s}(z, \sigma_q w) du - \frac{4\pi}{(s-1) \operatorname{vol}_{\text{hyp}}(X)} \right) + \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)}. \end{aligned} \quad (39)$$

For z and $w = u + iv \in X$ with $y_p > v$ and $vy_p > 1$, combining equations (33) and (39), we find that the right-hand side of the above equation decomposes into the following expression

$$\lim_{s \rightarrow 1} \left(\frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par},q}(z, s) - \frac{4\pi}{(s-1) \operatorname{vol}_{\text{hyp}}(X)} \right) + 4\pi \delta_{p,q}(v - y_p) + \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)}.$$

To evaluate the above limit, we compute the Laurent expansions of $\mathcal{E}_{\text{par},p}(w, s)$, $\operatorname{Im}(\sigma_p^{-1} z)^{1-s}$, and $(2s-1)^{-1}$ at $s = 1$. The Laurent expansions of $\operatorname{Im}(\sigma_p^{-1} z)^{1-s}$ and $(2s-1)^{-1}$ at $s = 1$ are easy to compute, and are of the form

$$\operatorname{Im}(\sigma_p^{-1} z)^{1-s} = 1 - (s-1) \log(\operatorname{Im}(\sigma_p^{-1} z)) + O((s-1)^2); \quad \frac{1}{2s-1} = 1 - 2(s-1) + O((s-1)^2).$$

Combining the above two equations with equation (5), we find

$$\begin{aligned} 4\pi \lim_{s \rightarrow 1} \left(\frac{\operatorname{Im}(\sigma_p^{-1} z)^{1-s}}{2s-1} \mathcal{E}_{\text{par},p}(w, s) - \frac{1}{(s-1) \operatorname{vol}_{\text{hyp}}(X)} \right) = \\ 4\pi \kappa_p(w) - \frac{8\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \frac{4\pi \log(\operatorname{Im}(\sigma_p^{-1} z))}{\operatorname{vol}_{\text{hyp}}(X)}, \end{aligned} \quad (40)$$

Combining the above computation with equation (39), we arrive at

$$\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du = 4\pi \kappa_q(z) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\operatorname{vol}_{\text{hyp}}(X)} + 4\pi \delta_{p,q}(v - y_p),$$

which proves equation (37).

We now prove equation (38). For $v > y_p$ and $vy_p > 1$, combining equations (34) and (39), we find

$$\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du = \lim_{s \rightarrow 1} \left(\frac{4\pi v^{1-s}}{2s-1} \mathcal{E}_{\text{par},q}(z, s) - \frac{4\pi}{(s-1) \text{vol}_{\text{hyp}}(X)} \right) + \frac{4\pi}{\text{vol}_{\text{hyp}}(X)}. \quad (41)$$

Combining equations (41) and (40), we find

$$\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du = 4\pi \kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)},$$

which proves equation (38), and hence, completes the proof of the lemma. \square

Proposition 2.3. *Let $p \in \mathcal{P}$ be a cusp. For $z, w = u + iv \in X$ with $y_p > 1$, we have the formal decomposition*

$$\begin{aligned} \int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P(w) \mu_{\text{hyp}}(w) &= \sum_{q \in \mathcal{P}} \int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2} + \\ &\sum_{q \in \mathcal{P}} \int_{1/y_p}^\infty \left(4\pi \kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + \\ &4\pi \int_{1/y_p}^{y_p} (v - y_p) \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v^2}. \end{aligned} \quad (42)$$

Proof. As the series $\Delta_{\text{hyp}} P(w)$ is absolutely and uniformly convergent, we have

$$\int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P(w) \mu_{\text{hyp}}(w) = \sum_{q \in \mathcal{P}} \sum_{\eta \in \Gamma_q \backslash \Gamma} \int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P_{\text{gen},q}(\eta w) \mu_{\text{hyp}}(w), \quad (43)$$

After making the substitution $w \mapsto \eta^{-1} \sigma_q w$, from the Γ -invariance of $g_{\text{hyp}}(z, w)$, and from the $\text{PSL}_2(\mathbb{R})$ -invariance of $\mu_{\text{hyp}}(z)$, formally for $w = u + iv \in X$, we find

$$\begin{aligned} \sum_{q \in \mathcal{P}} \sum_{\eta \in \Gamma_q \backslash \Gamma} \int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P_{\text{gen},q}(\eta w) \mu_{\text{hyp}}(w) &= \\ \sum_{q \in \mathcal{P}} \int_0^\infty \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2}. \end{aligned} \quad (44)$$

Recall from equation (24), that for any $w = u + iv \in \mathbb{H}$, the function $P_{\text{gen},q}(\sigma_q w)$ does not depend on u . So the right-hand side of equation (44) further decomposes to give

$$\begin{aligned} \sum_{q \in \mathcal{P}} \int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2} &+ \sum_{q \in \mathcal{P}} \int_{1/y_p}^{y_p} \left(\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du \right) \times \\ \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} &+ \sum_{q \in \mathcal{P}} \int_{y_p}^\infty \left(\int_0^1 g_{\text{hyp}}(z, \sigma_q w) du \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2}. \end{aligned} \quad (45)$$

Since in the second line of formula (45) we have $1/y_p < v < y_p$, we can apply equation (37), and rewrite the second line of formula (45) as

$$\begin{aligned} \sum_{q \in \mathcal{P}} \int_{1/y_p}^{y_p} \left(4\pi \kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} &+ \\ 4\pi \int_{1/y_p}^{y_p} (v - y_p) \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v^2}. \end{aligned} \quad (46)$$

Since in the third line of formula (45) we have $v > y_p > 1/y_p$, we can apply equation (38), and rewrite the third line of formula (45) as

$$\sum_{q \in \mathcal{P}} \int_{y_p}^\infty \left(4\pi \kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2}. \quad (47)$$

The proof of the proposition follows from combining equations (46) and (47). \square

Remark 2.4. The formal unfolding of the integral obtained in Proposition 2.3 translates into an equality of integrals, only if each of the three integrals on the right-hand side of equation (42) converges absolutely, which we prove in the lemmas that follow.

Lemma 2.5. *Let $p, q \in \mathcal{P}$ be two cusps. For $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2}$$

converges absolutely. Furthermore as $z \in X$ approaches a cusp $p \in \mathcal{P}$, we have

$$\sum_{q \in \mathcal{P}} \int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2} = o_z(1), \quad (48)$$

where the contribution from the term $o_z(1)$ is a smooth function in z , which approaches zero, as $z \in X$ approaches the cusp $p \in \mathcal{P}$.

Proof. For $v \in \mathbb{R}_{>0}$, from the formula for the function $\Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w)$ from equation 24, we derive that

$$\frac{\Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w)}{v^2} = \frac{8\pi^2}{\sinh^2(2\pi v)} - \frac{2}{v^2}$$

remains bounded. So it suffices to show that the integral

$$\int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) dudv$$

converges absolutely. Let \mathcal{I} denote the set $[0, 1] \times [0, 1/y_p]$. We view the above integral as a real-integral on the compact subset $\mathcal{I} \subset \mathbb{R}^2$. The hyperbolic Green's function $g_{\text{hyp}}(z, \sigma_q w)$ is at most log-singular on a measure zero subset of the interior points of \mathcal{I} . Furthermore from equation (20), the hyperbolic Green's function $g_{\text{hyp}}(z, \sigma_q w)$ is at most log log-singular on a measure zero subset of the boundary points of \mathcal{I} . Hence, it is absolutely integrable on \mathcal{I} . This implies that the integral

$$\int_0^{1/y_p} \int_0^1 g_{\text{hyp}}(z, \sigma_q w) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dudv}{v^2}$$

converges absolutely, and also proves the asymptotic relation asserted in equation (48). \square

Lemma 2.6. *Let $p, q \in \mathcal{P}$ be two cusps. For $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_{1/y_p}^{\infty} \left(4\pi\kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} \quad (49)$$

converges absolutely. Furthermore, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have

$$\begin{aligned} & \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \left(4\pi\kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} = \\ & 16\pi^2 \left(-y_p + \frac{|\mathcal{P}|(\log y_p + 1)}{\text{vol}_{\text{hyp}}(X)} - \sum_{q \in \mathcal{P}} k_{q,p}(0) + \frac{2\pi}{3} \right) + O\left(\frac{\log y_p}{y_p}\right). \end{aligned}$$

Proof. Substituting the formula for the function $\Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w)$ from equation (23), we have

$$\begin{aligned} & \int_{1/y_p}^{\infty} \left(4\pi\kappa_q(z) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \right) \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} = \\ & \left(8\pi\kappa_q(z) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \right) \int_{1/y_p}^{\infty} \left(\left(\frac{2\pi v}{\sinh(2\pi v)} \right)^2 - 1 \right) \frac{dv}{v^2}. \end{aligned}$$

The integral on the right-hand side of the above equation further simplifies to give

$$\begin{aligned} & \left(8\pi\kappa_q(z) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \right) \left[\frac{1}{v} - 2\pi \coth(2\pi v) \right]_{1/y_p}^{\infty} = \\ & \left(8\pi\kappa_q(z) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \right) \left(-2\pi - y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \right). \end{aligned} \quad (50)$$

Hence, from equation (50), we can conclude that the integral (49) converges absolutely.

We now compute the asymptotics of the expression obtained on the right-hand side of equation (50), as $z \in X$ approaches the cusp $p \in \mathcal{P}$. We first compute the asymptotics for the expression in the second bracket on the right-hand side of equation (50).

For $t \in \mathbb{R}_{>0}$, recall that the Taylor series expansion of the function $\coth(t)$ as t approaches zero is of the form

$$\coth(t) = \frac{1}{t} + \frac{t}{3} + O(t^3).$$

As $z \in X$ approaches $p \in \mathcal{P}$, the quantity $1/y_p$ approaches zero. So as $z \in X$ approaches $p \in \mathcal{P}$, using the Taylor expansion of $\coth(2\pi/y_p)$, we have the asymptotic relation

$$-2\pi - y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) = -2\pi - y_p + 2\pi \left(\frac{y_p}{2\pi} + \frac{2\pi}{3y_p} + O\left(\frac{1}{y_p^3}\right) \right) = -2\pi + \frac{4\pi^2}{3y_p} + O\left(\frac{1}{y_p^3}\right). \quad (51)$$

As $z \in X$ approaches $p \in \mathcal{P}$, from the Fourier expansion of Kronecker's limit function $\kappa_q(z)$ described in equation (6), we have the following asymptotic relation

$$8\pi\kappa_q(z) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} = 8\pi\delta_{p,q}y_p - \frac{8\pi \log y_p}{\text{vol}_{\text{hyp}}(X)} + 8\pi k_{q,p}(0) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} + O(e^{-2\pi y_p}). \quad (52)$$

Combining equations (51) and (52), as $z \in X$ approaches $p \in \mathcal{P}$, we have the asymptotic relation for the right-hand side of equation (50)

$$\begin{aligned} & \left(8\pi\delta_{p,q}y_p - \frac{8\pi \log y_p}{\text{vol}_{\text{hyp}}(X)} + 8\pi k_{q,p}(0) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} + O(e^{-2\pi y_p}) \right) \left(-2\pi + \frac{4\pi^2}{3y_p} + O\left(\frac{1}{y_p^3}\right) \right) = \\ & 16\pi^2 \left(-\delta_{p,q}y_p + \frac{(\log y_p + 1)}{\text{vol}_{\text{hyp}}(X)} - k_{q,p}(0) + \frac{2\pi}{3}\delta_{p,q} + O\left(\frac{\log y_p}{y_p}\right) \right). \end{aligned}$$

Hence, taking the summation over all $q \in \mathcal{P}$ completes the proof of the lemma. \square

Lemma 2.7. *Let $p, q \in \mathcal{P}$ be two cusps. For $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_{1/y_p}^{\infty} \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} \quad (53)$$

converges absolutely. Furthermore, we have the upper bound

$$\sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \left| \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq \frac{8\pi|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} \left(1 + \frac{4\pi^2}{3} \right). \quad (54)$$

Proof. We prove the upper bound asserted in (54), which also proves the absolute convergence of the integral in (53). Observing the elementary estimate

$$\begin{aligned} & \int_{1/y_p}^{\infty} \left| \log v \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq \\ & \int_0^1 \left| \log v \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} + \int_1^{\infty} \left| \log v \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2}, \end{aligned} \quad (55)$$

we proceed to bound the two integrals on the right-hand side of the above inequality. For $v \in \mathbb{R}_{>0}$, from the equation (23), we find that the function

$$-\frac{\Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w)}{v^2} = \frac{2}{v^2} - \frac{8\pi^2}{\sinh^2(2\pi v)}$$

is a positive monotone decreasing function, and hence, attains its maximum value at $v = 0$. So we compute the limit

$$-\lim_{v \rightarrow 0} \frac{\Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w)}{v^2} = \lim_{v \rightarrow 0} \left(\frac{2}{v^2} - \frac{8\pi^2}{\sinh^2(2\pi v)} \right) = \frac{8\pi^2}{3}.$$

So using the fact that, for $v \in (0, 1]$, $|\log v| = -\log v$, we have the following upper bound for the first integral on the right-hand side of inequality (55)

$$\int_0^1 \left| \log v \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq -\frac{8\pi^2}{3} \int_0^1 \log v dv = \frac{8\pi^2}{3}. \quad (56)$$

Again using formula (23), we derive

$$\max_{v \in \mathbb{R}_{>0}} \left| \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| = \max_{v \in \mathbb{R}_{>0}} \left(2 - \frac{8\pi^2 v^2}{\sinh^2(2\pi v)} \right) = 2.$$

Using the above bound, we derive the following upper bound for the second integral on the right-hand side of inequality (55)

$$\int_1^\infty \left| \log v \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq 2 \int_1^\infty \frac{\log v}{v^2} dv = 2 \left(\left[-\frac{\log v}{v} \right]_1^\infty + \left[-\frac{1}{v} \right]_1^\infty \right) = 2. \quad (57)$$

Hence, combining the upper bounds derived in equations (56) and (57) proves the lemma. \square

Lemma 2.8. *Let $p \in \mathcal{P}$ be a cusp. For $z \in X$ and $w = u + iv \in \mathbb{H}$ with $y_p > 1$, the integral*

$$-4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v^2} \quad (58)$$

converges absolutely. Furthermore, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have

$$-4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v^2} = 4\pi \left(4\pi y_p \coth(2\pi y_p) - 2 - \frac{8\pi^2}{3} \right) + O\left(\frac{1}{y_p^2}\right). \quad (59)$$

Proof. From equation (23), for a cusp $p \in \mathcal{P}$, we find

$$\begin{aligned} -4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v^2} &= -8\pi y_p \int_{1/y_p}^{y_p} \left(\frac{4\pi^2}{\sinh^2(2\pi v)} - \frac{1}{v^2} \right) dv = \\ &= -8\pi y_p \left[\frac{1}{v} - 2\pi \coth(2\pi v) \right]_{1/y_p}^{y_p} = -8\pi y_p \left(\frac{1}{y_p} - 2\pi \coth(2\pi y_p) - y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \right) = \\ &= -8\pi + 16\pi^2 y_p \coth(2\pi y_p) - 8\pi y_p \left(-y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \right). \end{aligned} \quad (60)$$

This implies that the integral (58) converges absolutely.

As $z \in X$ approaches the cusp $p \in \mathcal{P}$, from the Taylor expansion of $\coth(2\pi/y_p)$ already used in equation (51), we get

$$-8\pi y_p \left(-y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \right) = -8\pi y_p \left(\frac{4\pi^2}{3y_p} + O\left(\frac{1}{y_p^3}\right) \right) - \frac{32\pi^3}{3} + O\left(\frac{1}{y_p^2}\right),$$

which together with equation (60) completes the proof of the lemma. \square

Lemma 2.9. *Let $p \in \mathcal{P}$ be a cusp. For $z \in X$ and $w = u + iv \in \mathbb{H}$ with $y_p > 1$, the integral*

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v}. \quad (61)$$

converges absolutely. Furthermore, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v} = -8\pi \log y_p + 8\pi(1 - \log(4\pi)) + O\left(\frac{1}{y_p}\right). \quad (62)$$

Proof. Using equation (23), for a cusp $p \in \mathcal{P}$, we find

$$\begin{aligned} \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v} &= 2 \int_{1/y_p}^{y_p} \left(-\frac{1}{v} + \frac{4\pi^2 v}{\sinh^2(2\pi v)} \right) dv = \\ &= 2 \left[-\log v - 2\pi v \coth(2\pi v) + \log(\sinh(2\pi v)) \right]_{1/y_p}^{y_p}. \end{aligned}$$

Substituting the formulae for $\coth(2\pi v)$ and $\sinh(2\pi v)$, the right-hand side of the above equation can be further simplified to

$$2 \left[-\log v - 4\pi v - \frac{4\pi v}{e^{4\pi v} - 1} + \log \left(\frac{e^{4\pi v} - 1}{2} \right) \right]_{1/y_p}^{y_p}.$$

Observe that

$$\begin{aligned} &\left[-\log v - 4\pi v - \frac{4\pi v}{e^{4\pi v} - 1} + \log \left(\frac{e^{4\pi v} - 1}{2} \right) \right]_{1/y_p}^{y_p} = -\log y_p - 4\pi y_p - \frac{4\pi y_p}{e^{4\pi y_p} - 1} + \\ &\log(e^{4\pi y_p} - 1) + \log\left(\frac{1}{y_p}\right) + \frac{4\pi}{y_p} + \frac{4\pi}{y_p(e^{4\pi/y_p} - 1)} - \log(e^{4\pi/y_p} - 1) = \\ &-\log y_p - \log\left(\frac{e^{4\pi y_p}}{e^{4\pi y_p} - 1}\right) - \frac{4\pi y_p}{e^{4\pi y_p} - 1} + \frac{4\pi}{y_p} + \frac{4\pi}{y_p(e^{4\pi/y_p} - 1)} - \log(y_p(e^{4\pi/y_p} - 1)), \end{aligned} \quad (63)$$

which proves that the integral (61) converges absolutely.

We now compute the asymptotic expansion of each of the terms in the above expression, as $z \in X$ approaches the cusp $p \in \mathcal{P}$. As $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have the asymptotic relation for the first and second terms of (63)

$$-\log y_p - \log\left(\frac{e^{4\pi y_p}}{e^{4\pi y_p} - 1}\right) = -\log y_p + O(e^{-4\pi y_p}); \quad (64)$$

the third and fourth terms of (63) satisfy the asymptotic relation

$$-\frac{4\pi y_p}{e^{4\pi y_p} - 1} + \frac{4\pi}{y_p} = O\left(\frac{1}{y_p}\right); \quad (65)$$

the fifth term satisfies the asymptotic relation

$$\frac{4\pi}{y_p(e^{4\pi/y_p} - 1)} = \frac{4\pi}{y_p \left(\sum_{n=1}^{\infty} \frac{(4\pi)^n}{n! y_p^n} \right)} = 1 + O\left(\frac{1}{y_p}\right); \quad (66)$$

and the sixth term satisfies the asymptotic relation

$$\begin{aligned} -\log(y_p(e^{4\pi/y_p} - 1)) &= -\log\left(\sum_{n=1}^{\infty} \frac{(4\pi)^n}{n! y_p^{n-1}}\right) = \\ &= -\log\left(4\pi + \sum_{n=1}^{\infty} \frac{(4\pi)^{n+1}}{(n+1)! y_p^n}\right) = -\log(4\pi) + O\left(\frac{1}{y_p}\right). \end{aligned} \quad (67)$$

Substituting the asymptotic relations obtained in equations (64), (65), (66), and (67) into (63), we derive the asymptotic relation

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\text{hyp}} P_{\text{gen},p}(\sigma_p w) \frac{dv}{v} = -8\pi \log y_p + 8\pi(1 - \log(4\pi)) + O\left(\frac{1}{y_p}\right),$$

as $z \in X$ approaches the cusp $p \in \mathcal{P}$, which completes the proof of the lemma. \square

In the following proposition, combining all the asymptotics established in this section, we compute the asymptotics of the integral

$$\int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P(w) \mu_{\text{hyp}}(w),$$

as $z \in X$ approaches a cusp $p \in \mathcal{P}$.

Proposition 2.10. *Let $p \in \mathcal{P}$ be a cusp. Then, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have*

$$\begin{aligned} \int_X g_{\text{hyp}}(z, w) \Delta_{\text{hyp}} P(w) \mu_{\text{hyp}}(w) = & - \\ & \frac{32\pi^2(g-1)\log y_p}{\text{vol}_{\text{hyp}}(X)} - \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + \alpha_p + o_z(1), \\ \text{where } \alpha_p = & \frac{16\pi^2|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} - 16\pi^2 \sum_{q \in \mathcal{P}} k_{q,p}(0) - 8\pi \log(4\pi), \end{aligned} \quad (68)$$

and the contribution from the term $o_z(1)$ is a smooth function in z , which approaches zero, as $z \in X$ approaches the cusp $p \in \mathcal{P}$.

Proof. From Lemmas 2.5, 2.6, 2.7, 2.8, and 2.9, it follows that each of the integrals on the right-hand side of the equation (42) is absolutely convergent. This implies that the equality of integrals described in equation (42) indeed holds true for all $z \in X$ provided that $y_p > 1$.

As $z \in X$ approaches the cup $p \in \mathcal{P}$, combining Lemmas 2.5 and 2.6, we find that the first two integrals on the right-hand side of equation (42) yield

$$\begin{aligned} & 16\pi^2 \left(-y_p + \frac{|\mathcal{P}|(\log y_p + 1)}{\text{vol}_{\text{hyp}}(X)} - \sum_{q \in \mathcal{P}} k_{q,p}(0) + \frac{2\pi}{3} \right) - \\ & \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + o_z(1), \end{aligned} \quad (69)$$

where the contribution from the term $o_z(1)$ is a smooth function in z , which approaches zero, as $z \in X$ approaches $p \in \mathcal{P}$. As $z \in X$ approaches the cusp $p \in \mathcal{P}$, combining Lemmas 2.8 and 2.9, we find that the third integral on the right-hand side of equation (42) yields

$$16\pi^2 y_p \coth(2\pi y_p) - 8\pi \log y_p - \frac{32\pi^3}{3} - 8\pi \log(4\pi) + O\left(\frac{1}{y_p}\right). \quad (70)$$

Combining (69) and (70), as $z \in X$ approaches the cusp $p \in \mathcal{P}$, the right-hand side of equation (42) simplifies to

$$\begin{aligned} & -16\pi^2 y_p + 16\pi^2 y_p \coth(2\pi y_p) + \frac{16\pi^2|\mathcal{P}|\log y_p}{\text{vol}_{\text{hyp}}(X)} - 8\pi \log y_p + \frac{16\pi^2|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} - \\ & 16\pi^2 \sum_{q \in \mathcal{P}} k_{q,p}(0) - \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} - 8\pi \log(4\pi) + o_z(1). \end{aligned} \quad (71)$$

As $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have the following asymptotic relation for the first two terms in the above expression

$$16\pi^2 y_p (\coth(2\pi y_p) - 1) = 16\pi^2 y_p \left(\frac{\cosh(2\pi y_p) - \sinh(2\pi y_p)}{\sinh(2\pi y_p)} \right) = O(e^{-y_p}).$$

Furthermore, as $z \in X$ approaches the cusp $p \in \mathcal{P}$ the third and fourth terms in expression (71) give

$$\frac{16\pi^2|\mathcal{P}|\log y_p}{\text{vol}_{\text{hyp}}(X)} - 8\pi\log y_p = -\frac{32\pi^2(g-1)\log y_p}{\text{vol}_{\text{hyp}}(X)}.$$

Hence, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, the expression in (71) further reduces to give

$$\begin{aligned} & -\frac{32\pi^2(g-1)\log y_p}{\text{vol}_{\text{hyp}}(X)} - \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + \\ & \frac{16\pi^2|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} - 16\pi^2 \sum_{q \in \mathcal{P}} k_{q,p}(0) - 8\pi \log(4\pi) + o_z(1), \end{aligned}$$

which completes the proof of the proposition. \square

Corollary 2.11. *Let $p \in \mathcal{P}$ be a cusp. Then, as $z \in X$ approaches the cusp $p \in \mathcal{P}$, we have*

$$\begin{aligned} \phi(z) = & -\frac{4\pi \log y_p}{\text{vol}_{\text{hyp}}(X)} - \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log v}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + \\ & \frac{\alpha_p}{8\pi g} + \frac{2\pi k_{p,p}(0)}{g} - \frac{C_{\text{hyp}}}{8g^2} - \frac{2\pi c_X}{g \text{vol}_{\text{hyp}}(X)} + o_z(1), \end{aligned}$$

where the constant α_p is as defined in (68), and the contribution from the term $o_z(1)$ is a smooth function in z , which approaches zero, as $z \in X$ approaches the cusp $p \in \mathcal{P}$.

Proof. As $z \in X$ approaches the cusp $p \in \mathcal{P}$, from equation (27), we have

$$\frac{H(z)}{2g} = -\frac{4\pi \log y_p}{g \text{vol}_{\text{hyp}}(X)} - \frac{2\pi}{g \text{vol}_{\text{hyp}}(X)} + \frac{2\pi k_{p,p}(0)}{g} + O\left(\frac{1}{y_p}\right). \quad (72)$$

Furthermore, from Proposition 2.10, we find that

$$\begin{aligned} & \frac{1}{8\pi g} \int_X g_{\text{hyp}}(z, \zeta) \Delta_{\text{hyp}} P(\zeta) \mu_{\text{hyp}}(\zeta) = -\frac{4\pi \log y_p}{\text{vol}_{\text{hyp}}(X)} + \frac{4\pi \log y_p}{g \text{vol}_{\text{hyp}}(X)} - \\ & \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log v}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},q}(\sigma_q w) \frac{dv}{v^2} + \frac{\alpha_p}{8\pi g} + o_z(1), \end{aligned} \quad (73)$$

where the contribution from the term $o_z(1)$ is a smooth function in z , which approaches zero, as $z \in X$ approaches the cusp $p \in \mathcal{P}$. The proof of the corollary follows from combining equations (30), (72), and (73). \square

The following proposition has been proved as Proposition 6.1.9 in [4] (or Proposition 4.10 in [3]). However, for the convenience of the reader, we reproduce the proof here.

Proposition 2.12. *We have the following upper bound*

$$\frac{|C_{\text{hyp}}|}{8g^2} \leq \frac{2\pi (d_X + 1)^2}{\lambda_1 \text{vol}_{\text{hyp}}(X)},$$

where λ_1 denotes the first non-zero eigenvalue of the hyperbolic Laplacian acting on smooth functions defined on X .

Proof. Recall that C_{hyp} is defined as

$$C_{\text{hyp}} = \int_X \int_X g_{\text{hyp}}(\zeta, \xi) \left(\int_0^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \left(\int_0^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; \xi) dt \right) \mu_{\text{hyp}}(\xi) \mu_{\text{hyp}}(\zeta).$$

From formulae (29), (30), we have

$$\begin{aligned}\Delta_{\text{hyp}} \phi(z) &= \frac{4\pi \mu_{\text{can}}(z)}{\mu_{\text{hyp}}(z)} - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \implies \int_X \Delta_{\text{hyp}} \phi(z) \mu_{\text{hyp}}(z) = 0, \\ \phi(z) &= \frac{1}{2g} \int_X g_{\text{hyp}}(z, \zeta) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g^2},\end{aligned}\tag{74}$$

respectively. So combining the above two equations, we get

$$-\frac{1}{4\pi} \int_X \phi(z) \Delta_{\text{hyp}} \phi(z) \mu_{\text{hyp}}(z) = \frac{1}{2g} \int_X \int_X g_{\text{hyp}}(z, \zeta) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) \mu_{\text{can}}(z).\tag{75}$$

Observe that

$$\int_X g_{\text{hyp}}(z, \zeta) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) = 2g\phi(z) + \frac{C_{\text{hyp}}}{4g} \in C_{\ell, \ell}(X).$$

So using equations (31) and (75), we derive

$$\begin{aligned}\int_X \phi(z) \Delta_{\text{hyp}} \phi(z) \mu_{\text{hyp}}(z) &= \frac{\pi}{g^2} \int_X \int_X g_{\text{hyp}}(z, \zeta) \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \times \\ &\quad \left(\int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(\zeta) \mu_{\text{hyp}}(z) = \frac{\pi C_{\text{hyp}}}{g^2}.\end{aligned}\tag{76}$$

From equation (74), we have

$$\sup_{z \in X} |\Delta_{\text{hyp}} \phi(z)| \leq \sup_{z \in X} \left| \frac{4\pi \mu_{\text{can}}(z)}{\text{vol}_{\text{hyp}}(X) \mu_{\text{shyp}}(z)} \right| + \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} = \frac{4\pi(d_X + 1)}{\text{vol}_{\text{hyp}}(X)},\tag{77}$$

where d_X is as defined in (2). As the function $\phi(z) \in L^2(X)$, it admits a spectral expansion in terms of the eigenfunctions of the hyperbolic Laplacian Δ_{hyp} . So from the arguments used to prove Proposition 4.1 in [7], we have

$$\left| \int_X \phi(z) \Delta_{\text{hyp}} \phi(z) \mu_{\text{hyp}}(z) \right| \leq \sup_{z \in X} \frac{|\Delta_{\text{hyp}} \phi(z)|^2}{\lambda_1} \int_X \mu_{\text{hyp}}(z),\tag{78}$$

where λ_1 denotes the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} . Hence, from equation (76), and combining estimates (77) and (78), we arrive at the estimate

$$|C_{\text{hyp}}| = \frac{g^2}{\pi} \left| \int_X \phi(z) \Delta_{\text{hyp}} \phi(z) \mu_{\text{hyp}}(z) \right| \leq \frac{g^2}{\pi \lambda_1} \int_X |\Delta_{\text{hyp}} \phi(z)|^2 \mu_{\text{hyp}}(z) \leq \frac{16\pi g^2 (d_X + 1)^2}{\lambda_1 \text{vol}_{\text{hyp}}(X)},$$

which completes the proof of the proposition. \square

Theorem 2.13. *Let $p, q \in \mathcal{P}$ be two cusps with $p \neq q$. Then, we have the upper bound*

$$\begin{aligned}|g_{\text{can}}(p, q)| &\leq 4\pi |k_{p,q}(0)| + \frac{2\pi}{g} \left(\sum_{\substack{s \in \mathcal{P} \\ s \neq p}} |k_{s,p}(0)| + \sum_{\substack{s \in \mathcal{P} \\ s \neq q}} |k_{s,q}(0)| \right) + \\ &\quad \frac{1}{\text{vol}_{\text{hyp}}(X)} \left(\frac{4\pi(d_X + 1)^2}{\lambda_1} + \frac{|4\pi c_X|}{g} + \frac{43|\mathcal{P}|}{g} + 4\pi \right) + \frac{2 \log(4\pi)}{g}.\end{aligned}$$

Proof. For $z, w \in X$, from equation (29), we have

$$g_{\text{can}}(p, q) = \lim_{z \rightarrow p} \lim_{w \rightarrow q} (g_{\text{hyp}}(z, w) - \phi(z) - \phi(w)).$$

Combining equation (19) with Corollary 2.11, for a fixed $w \in X$ with $z \in X$ approaching the cusp $p \in \mathcal{P}$, we have

$$\begin{aligned} \lim_{z \rightarrow p} (g_{\text{hyp}}(z, w) - \phi(z)) &= 4\pi\kappa_p(w) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{\alpha_p}{8\pi g} - \frac{2\pi k_{p,p}(0)}{g} + \\ &\frac{C_{\text{hyp}}}{8g^2} + \frac{2\pi c_X}{g \text{vol}_{\text{hyp}}(X)} + \lim_{y_p \rightarrow \infty} \sum_{s \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log \zeta}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},s}(\sigma_s \xi) \frac{d\zeta}{\zeta^2}, \end{aligned} \quad (79)$$

where $\zeta = \text{Im}(\xi)$. As $w \in X$ approaches the cusp $q \in \mathcal{P}$ with $q \neq p$, from the Fourier expansion of the Kronecker's limit function $\kappa_p(w)$, stated in equation 6, we have

$$4\pi\kappa_p(w) = 4\pi k_{p,q}(0) - \frac{4\pi \log v_q}{\text{vol}_{\text{hyp}}(X)} + O(e^{-2\pi v_q}).$$

So using Corollary 2.11 one more time, and substituting the above asymptotic relation into equation (79), we compute the limit

$$\begin{aligned} \lim_{z \rightarrow p} \lim_{w \rightarrow q} (g_{\text{hyp}}(z, w) - \phi(z) - \phi(w)) &= 4\pi k_{p,q}(0) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{\alpha_p}{8\pi g} - \frac{2\pi k_{p,p}(0)}{g} - \\ &\frac{\alpha_q}{8\pi g} - \frac{2\pi k_{q,q}(0)}{g} + \frac{C_{\text{hyp}}}{4g^2} + \frac{4\pi c_X}{g \text{vol}_{\text{hyp}}(X)} + \lim_{y_p \rightarrow \infty} \sum_{s \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log \zeta}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},s}(\sigma_s \xi) \frac{d\zeta}{\zeta^2} + \\ &\lim_{v_q \rightarrow \infty} \sum_{s \in \mathcal{P}} \int_{1/v_q}^{\infty} \frac{\log \zeta}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},s}(\sigma_s \xi) \frac{d\zeta}{\zeta^2}. \end{aligned} \quad (80)$$

Using the definition of the constant α_p from (68), we find that the first six terms on the right-hand side of the above equation give

$$\begin{aligned} &4\pi k_{p,q}(0) - \frac{1}{g} \left(\frac{2\pi|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} - 2\pi \sum_{s \in \mathcal{P}} k_{s,p}(0) - \log(4\pi) \right) - \frac{2\pi k_{p,p}(0)}{g} - \\ &\frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \frac{1}{g} \left(\frac{2\pi|\mathcal{P}|}{\text{vol}_{\text{hyp}}(X)} - 2\pi \sum_{s \in \mathcal{P}} k_{s,q}(0) - \log(4\pi) \right) - \frac{2\pi k_{q,q}(0)}{g} = \\ &4\pi k_{p,q}(0) + \frac{2\pi}{g} \left(\sum_{\substack{s \in \mathcal{P} \\ s \neq p}} k_{s,p}(0) + \sum_{\substack{s \in \mathcal{P} \\ s \neq q}} k_{s,q}(0) \right) - \frac{4\pi(|\mathcal{P}| + g)}{g \text{vol}_{\text{hyp}}(X)} + \frac{2\log(4\pi)}{g}. \end{aligned}$$

Furthermore, the expression on the right-hand side of the above equation can be bounded by

$$4\pi |k_{p,q}(0)| + \frac{2\pi}{g} \left(\sum_{\substack{s \in \mathcal{P} \\ s \neq p}} |k_{s,p}(0)| + \sum_{\substack{s \in \mathcal{P} \\ s \neq q}} |k_{s,q}(0)| \right) + \frac{13|\mathcal{P}| + 4\pi g}{g \text{vol}_{\text{hyp}}(X)} + \frac{2\log(4\pi)}{g}. \quad (81)$$

Using Proposition 2.12, we derive the upper bound for the next two terms on the right-hand side of equation (80)

$$\frac{C_{\text{hyp}}}{4g^2} + \frac{4\pi c_X}{g \text{vol}_{\text{hyp}}(X)} \leq \frac{4\pi (d_X + 1)^2}{\lambda_1 \text{vol}_{\text{hyp}}(X)} + \frac{|4\pi c_X|}{g \text{vol}_{\text{hyp}}(X)}. \quad (82)$$

From Lemma 2.7, we have the upper bound for the absolute value of the last two terms on the right-hand side of equation (80)

$$\begin{aligned} &\lim_{y_p \rightarrow \infty} \sum_{s \in \mathcal{P}} \int_{1/y_p}^{\infty} \left| \frac{\log \zeta}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},s}(\sigma_s \xi) \right| \frac{d\zeta}{\zeta^2} + \\ &\lim_{v_q \rightarrow \infty} \sum_{s \in \mathcal{P}} \int_{1/v_q}^{\infty} \left| \frac{\log \zeta}{2g \text{vol}_{\text{hyp}}(X)} \Delta_{\text{hyp}} P_{\text{gen},s}(\sigma_s \xi) \right| \frac{d\zeta}{\zeta^2} \leq \frac{2|\mathcal{P}|}{g \text{vol}_{\text{hyp}}(X)} \left(1 + \frac{4\pi^2}{3} \right) \leq \frac{30|\mathcal{P}|}{g \text{vol}_{\text{hyp}}(X)}. \end{aligned} \quad (83)$$

The proof of the theorem follows from combining the estimates obtained in equations (81), (82), and (83). \square

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